4. Semidefinite program relaxations of integer programs.

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We continue the discussion on how to improve the initial relaxation made of an integer program. Here, we focus on semidefinite program (SDP) relaxations, and we introduce Lovász-Schrijver relaxation with semidefinite programming, and the Positivstellensatz, Positivstellensatz Calculus and sums-of-squares (SOS) proof systems.

http://www.csc.kth.se/~lauria/sos14/

Motivation for SDP relaxations

Semidefinite programming (SDP) relaxations are more complicated to construct than linear relaxations, but more powerful. To see this, we consider the maximum cut problem as an example.

Maximum cut problem[1]

We consider an undirected and loopless graph $G(V, E)$, where V denotes the set of vertexes and E denotes the set of edges. Each edge is weighted according to some function $c : E \rightarrow \mathbb{R}$.

Let $S \subseteq V$, the *cut* induced by S is defined as the edges with one end in S and the other end in $V \setminus S$ and the *weight of the cut* is the sum of the weights corresponding to the edges belonging to the cut. The maximum cut problem is the problem of finding the set S that maximizes the weight of the induced cut, and it is NP-complete[2].

Several approximation algorithms have been developed to be able to solve it with a high *performance guarantee*[3] in polynomial time. The approximation algorithm with the highest known performance guarantee is based on an SDP relaxation that achieves $\rho \approx 0.878$.

However, any algorithm based on Sherali-Adams achieves a performance guarantee of at most $1/2 + \epsilon$, even after a degree $t = n^{\delta(\epsilon)}$ relaxation.[4] Notice that it is very easy to achieve performance $1/2$ by a greedy approach.

[1] Poljak and Tuza. Maximum cuts and largest bipartite subgraphs. *Combinatorial Optimization*, 20, 1995

[2] Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*. Plenum Press, 1972

[3] The performance guarantee is defined as the maximum over all instances of

$$\rho = \frac{\text{value of the solution}}{\text{optimal value}} .$$

[4] Moses Charikar, Konstantin Makarychev, and Yury Makarychev. Integrality gaps for sherali-adams relaxations. In *Proceedings of the 41st annual ACM symposium on Theory of computing*, pages 283–292. ACM, 2009

Lovász-Schrijver hierarchy with SDP

The first SDP relaxation we discuss is the Lovász-Schrijver hierarchy with SDP (LS_+) which a lift-and-project relaxation similar to the Lovász-Schrijver hierarchy we discussed in the first and third lecture. As a proof system it can be seen as an inference system of linear and quadratic inequalities. There is a natural interpretation of rank of the inequalities derived. The initial linear inequalities have rank 0. Linear inequalities of rank $< r$ can be lifted (i.e. multiplied by either x_i or $(1 - x_i)$) to obtain quadratic inequalities of rank r, and and positively summed to axioms $x_i^2 - x_i \geq 0$, $x_i - x_i^2 \geq 0$ and other linear inequalities of rank $< r$ in order to obtain new linear inequalities of rank r.

The new proof system works in the same way as Lovász-Schrijver, but when deriving a linear inequality of rank r from inequalities of rank $< r$ it is possible to also sum quadratic inequalities $(p(x))^2 \geq 0$ for any polynomial p of degree 1.

Geometric interpretation

From the geometric point of view, the only difference with the constraints on the matrix Y imposed by Lovász-Schrijver seen in is that Y must be positive semidefinite, i.e. $Y \succeq 0$ is added.

As for the Lovász-Schrijver hierarchy, we introduce the auxiliary variable $x_0 = 1$ and perform the variable substitution $x_i x_j = y_{ij}$ for every $i, j = 0, \ldots, n$. We let any terms of degree zero be represented by y_{00}. We enforce the following constraints on Y:

1. $Y = Y^T$,
2. $y_{0i} = y_{i0} = y_{ii}, \quad \text{for } i = 0, \ldots, n,$
3. $Y^{(i)} \in P \quad \text{for } i = 0, \ldots, n,$
4. $Y^{(0)} - Y^{(i)} \in P \quad \text{for } i = 0, \ldots, n.$
5. $Y \succeq 0.$

Constraints 1 and 2 are the linearization of equations $x_i x_j = x_j x_i$ and $x_i - x_i^2 = 0$. Constraints 3 and 4 stem from

$$\sum_{j=1}^{n} a_j x_j x_i - b x_i \leq 0, \quad \text{for all } \sum_{j=1}^{n} a_j x_j \leq b \text{ defining } P,$$

and

$$\sum_{j=1}^{n} a_j x_j (1 - x_i) - b(1 - x_i) \leq 0, \quad \text{for all } \sum_{j=1}^{n} a_j x_j \leq b \text{ defining } P,$$

respectively. The matrix constraint 5 is equivalent to ask that $p^2 \geq 0$ for all linear polynomials p.

Operator and rank

The operation $N_+(P)$ is the projection over $1 = Y_{00}, x_1 = Y_{01}, \ldots, x_n = Y_{0n}$ of the solutions of the semidefinite program obtained by the relaxation. If P is the polytope corresponding to the set of initial inequalities then $N_+(P)$ is determined by inequalities derivable in rank one, and $N_+^t(P)$ is the polytope obtained by applying N_+ to the polytope $N_+^{t-1}(P)$ for $t > 1$, with the convention that $N_+^0(P) = P$. Notice that contrary to $N^t(P)$, the convex set $N_+^t(P)$ is not necessarily a polytope. Optimization over $N_+^t(P)$ can be done in $n^{\mathcal{O}(t)}$ given a polynomial time separator of P. We can use the essentially same separator as for $N^t(P)$, plus a separator for the constraint $Y \succeq 0$.

Example 1. Consider the maximum independent set problem on a graph $G(V,E)$

$$\begin{aligned} \text{maximize} \quad & \sum_{u \in V} x_u \\ \text{subject to} \quad & x_u + x_v \leq 1, \quad (u,v) \in E \\ & x_u \in \{0,1\}. \end{aligned} \tag{1}$$

Over the complete graph of order n we want to show that the objective value of (1) is at most one, this is equivalent to showing that $\sum x_u \leq 1$. If we are using LS relaxations, then we need rank n. Using LS$_+$, we can do this in rank one. The axioms of the proof are $1 - x_u - x_v \geq 0$, $x_u^2 - x_u = 0$, and we can derive

$$\begin{aligned} & \sum_u \sum_{v \neq u} (1 - x_u - x_v) x_u + \sum_u (x_u^2 - x_u)(n-2) + (1 - \sum_u x_u)^2 \geq 0 \\ & \Rightarrow 1 - \sum_u x_u \geq 0. \end{aligned}$$

Moment matrix[5]

[5] Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11:796–817, 2001

At every step of the Lovász-Schrijver hierarchy with SDP we define a matrix that is interpreted as the values of the degree two monomials. In this section we generalize this matrix to higher degrees and we see that positive semidefinite moment matrices are fundamental to show polynomial inequalities. We can express any polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$ of degree t as

$$p(x) = \sum_{\alpha \in [t]^n : \sum_i \alpha_i \leq t} p_\alpha x^\alpha, \text{ with } x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \text{ and } \sum_{i=1}^n \alpha_i \leq t,$$

where

$$1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_1x_n, \ldots, x_n^2, \ldots, x_1^t, \ldots, x_n^t \tag{2}$$

is a basis of $p(x)$ and p is the coefficient vector of $p(x)$ with respect to (2).

For every monomial of degree at most $2t$ we introduce the variable substitution $x_1^i x_2^j \cdots x_n^l = y_{ij\ldots l}$. The moment matrix $M_t(y)$ has rows and columns labelled according to the basis (2) of monomials of degree at most t, and its elements are equal to the variable $y_{ij\ldots l}$ that corresponds to the product of those monomials. For example, we show the case for $n = 2$ and $t = 2$,

$$M_2(y) = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\ \begin{pmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{30} & y_{22} & y_{13} & y_{04} \end{pmatrix} \end{array}.$$

We can use this moment matrix to express a linearization of polynomial $p(x)$, so that we can write polynomial equations and inequalities as matrix inequalities. Indeed for any polynomial p of degree $\leq t$ there is some **symmetric** matrix N_p so that $N_p \bullet M_{t/2}(y)$ is the linearized version[6] of $p(x) \geq 0$. For example $p(x) = x_1^3 x_2 + 3x_1x_2$ of degree 4 linearizes into $y_{31} + 3y_{11}$. Let N_p be the matrix with the same rows and columns of $M_{2(y)}$, with

- $1/2$ in the entry indexed by row x_1x_2 and column x_1^2,
- $1/2$ in the entry indexed by row x_1^2 and column x_1x_2,
- $3/2$ in the entry indexed by row x_1 and column x_2,
- $3/2$ in the entry indexed by row x_2 and column x_1,
- 0 everywhere else.

[6] A linearization of a polynomial is the substitution of its monomials with new formal variable.

Clearly $N_p \bullet M_{t/2}(y) = y_{31} + 3y_{11}$.

Given a polynomial p of degree t, its square is the polynomial

$$p^2(x) = (\sum_\alpha p_\alpha x^\alpha)^2 = \sum_\alpha \sum_{\alpha'} p_\alpha x^\alpha p_{\alpha'} x^{\alpha'}.$$

The linearization of the inequality $p^2(x) \geq 0$ can easily be expressed using the moment matrix $M_t(y)$ and defining N_p as pp^T, where p is identified with its vector of coefficients.

$$pp^T \bullet M_t(y) = p^T M_t(y) p \geq 0\ .$$

The important point here is that unfortunately we cannot efficiently find a matrix $M_t(y)$ that enforces that the entries are consistent with the product between variables (e.g., we cannot enforce that $y_{02}y_{10} = y_{12}$). Nevertheless we can enforce that for every polynomial p of degree t the multilinearization of p^2, which is equal to $p^T M_t(y) p$, is non-negative. Luckily just by asking for $M_t(y)$ to be positive semidefinite we get $p^T M_t(y) p \geq 0$ for every p.

In Lovász-Schrijver with SDP we used the moment matrix Y for $t = 1$. We will use the moment matrix again to turn systems of a polynomial equations and inequalities into SDP program. This is a relaxation (because the integer solutions are still valid) and retains the constraint that a squared polynomial must positive.

Interlude: Sums-of-squares and polynomial inequalities

Let us take a break from relaxations and just focus on one method for proving polynomial inequalities. We want to show that a polynomial p is non-negative over $\mathbb{R}^n$, and a way to do that is to show that p is expressible as a sum of squared polynomials. If this representation exists, we can find it efficiently by reformulating the problem as a semidefinite program[7]. Let us see a small example.

[7] Pablo A Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical programming*, 96(2):293–320, 2003

$$F(x, y) = 2x^4 + 2x^3y - x^2y^2 + 5y^4$$

and assume it is expressible as a sum of squared polynomials. These have degree at least 2, half the degree of F, and furthermore since F is homogeneous they have degree exactly 2. The monomials in (x, y) of degree 2 are x^2, y^2

and xy. The square of a polynomial $a_1x^2 + a_2y^2 + a_3xy$ can be written as

$$\begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}^T \underbrace{\begin{pmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{pmatrix}}_{A} \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}$$

where A is by construction a rank 1 positive semidefinite matrix. Therefore the polynomial $F(x,y)$ is a sum of squared polynomials if and only if it can be written as the following matrix product

$$\begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}^T \underbrace{\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}}_{Q} \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix} =$$

$$q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3. \tag{3}$$

where

- Q is the sum of rank 1 positive semidefinite matrices (i.e., it is a positive semidefinite matrix);
- the coefficients in (3) must match $F(x,y)$, which means that the matrix Q must satisfy that

$$\begin{aligned} q_{11} &= 2 \\ q_{22} &= 5 \\ q_{13} &= 1 \\ q_{23} &= 0 \\ q_{33} + 2q_{12} &= -1 \end{aligned}$$

Finding a matrix $Q \succeq 0$ which satisfies a set of linear equations is exactly the goal of semidefinite programming. Let consider the following solution to the program,

$$Q = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix}.$$

To construct the sum of squared polynomials yielding $F(x,y)$, we perform a factorization on $Q = U^TU$ where the row of U are orthogonal,[8] and we get:

$$F(x,y) = \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}^T \begin{pmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix} =$$

$$\begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}^T \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 \\ -3 & 1 \\ 1 & 3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix} =$$

$$\left(\frac{1}{\sqrt{2}}(2x^2 - 3y^2 + xy)\right)^2 + \left(\frac{1}{\sqrt{2}}(y^2 + 3xy)\right)^2.$$

[8] Given the (efficiently computable) spectral decomposition $Q = W\Delta W^T$, where W is a unitary matrix and Δ is diagonal and non negative, we can set $U = W\sqrt{\Delta}$.

We have reformulated $F(x, y)$ as a sum of squared polynomials and, consequently, we have shown that $F(x, y) \geq 0$. It is important to note that not every polynomial $p \geq 0$ on $\mathbb{R}^n$ can be expressed as a sum of squared polynomials if $n \geq 2$. An example of this is the Motzkin polynomial[9],

$$M(x, y, z) = x^6 + y^4z^2 + y^2z^4 - 3x^2y^2z^2.$$

[9] Motzkin. The arithmetic-geometric inequality. In *Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965)*. Academic Press, 1967

Positivstellensatz proof system[10]

[10] Dima Grigoriev and Nicolai Vorobjov. Complexity of null-and positivstellensatz proofs. *Annals of Pure and Applied Logic*, 113(1):153–160, 2001

We consider now a system of polynomial equations and inequalities

$$f_1 = 0, \ldots, f_k = 0,\ h_1 \geq 0, \ldots, h_m \geq 0, \tag{4}$$

and we say that $p \geq 0$ has Positivstellensatz proof from (4) if we can write

$$p = \sum_{s=1}^{k} f_s g_s + \sum_{I \subseteq \{1,\ldots,m\}} \Big(\prod_{i \in I} h_i\Big)\Big(\sum_j e_{I,j}^2\Big), \tag{5}$$

where $e_{I,j}$ and g_s are arbitrary polynomials. It is easy to see that p is indeed non negative when the assumptions in (4) hold. That is to say that the proof system is sound. We will be study the degree of Positivstellensatz proofs as in (5), which is defined to be equal to

$$\max_{s,I,j}\{\deg(f_s, g_s), \deg(e_{I,j}^2 \prod_{i \in I} h_i)\}. \tag{6}$$

Exercise 2. Show that whatever can be proved using an assumption $f = 0$ can also be proved using the assumptions $f \geq 0$, $-f \geq 0$ instead.

Notice that a proof without assumptions is just a proof that the polynomial p can be written as a sum of squares. We know already from the Motzkin polynomial that there are polynomials that are non negative and yet they cannot written as a sum of squares, so the know that there are polynomial inequalities that cannot be proved using Positivstellensatz. Nevertheless it is always possible to find a refutation for systems of inequalities that are incompatible over the reals. This is definitely non-trivial, and it means that Positivstellensatz is a complete proof system.[11]

[11] Bochnak, Coste, and Roy. *Geometrie Algebrique Reelle*. Springer-verlag, 1987

Theorem 3 (Completeness of Positivstellensatz over $\mathbb{R}^n$)**.** *Consider a system of polynomial equations and inequalities*

$$f_1 = 0, \ldots, f_k = 0,\ h_1 \geq 0, \ldots, h_m \geq 0 \tag{7}$$

with no common solution over $\mathbb{R}^n$, then there esists a Positivstellensatz refutation of them. Namely we can write

$$-1 = \sum_{s=1}^{k} f_s g_s + \sum_{I \subseteq \{1,\ldots,m\}} \Big(\prod_{i \in I} h_i\Big)\Big(\sum_j e_{I,j}^2\Big), \tag{8}$$

where f_i and h_i are polynomials on $x_1, \ldots, x_n$.

Boolean setting. For most of our lectures we will discuss polynomials over variables with values in $\{0,1\}$. In that case we always assume, even if we don't always mention it, to have constraints

$$\begin{aligned} x_i &\geq 0 \\ 1 - x_i &\geq 0 \\ x_i^2 - x_i &= 0 \end{aligned}$$

in the axiom set. These constraints enforce the boolean setting. Actually it is possible to get rid of the two inequalities with a minor increase in the degree of the proof, using the following identities.

$$\begin{aligned} x_i &= -(x_i^2 - x_i) + x_i^2 \\ 1 - x_i &= -(x_i^2 - x_i) + (1 - x_i)^2 \ . \end{aligned}$$

It is worth to observe that in the boolean setting we don't just have completeness, but also *implicational completeness*. We will see later what this does means and also that this holds already for a subsystem of Positivstellensatz.

Positivstellensatz calculus[12]

Positivstellensatz calculus is a more powerful version of Positivstellensatz. Consider constraints as in (4), a proof that $p \geq 0$ in Positivstellensatz Calculus is the writing of

$$p = f + \sum_{I \subseteq \{1,\ldots,m\}} \Big(\prod_{i \in I} h_i\Big)\Big(\sum_j e_{I,j}^2\Big), \tag{9}$$

where f is an arbitrary polynomial derived from $f_1 = 0, \ldots, f_k = 0$ using a polynomial calculus derivation.[13] If $f + h = -1$, we call this a *Positivstellensatz calculus refutation* of (4). The degree of the refutation is the maximum among the degree of summands $e_{I,j}^2 \prod_{i \in I} h_i$ and the degree of the derivation of f. Any Positivstellensatz proof is also a Positivstellensatz Calculus proof, nevertheless it is possible that the latter proof system has a proof of smaller degree than the former proof system.

[12] Dima Grigoriev and Nicolai Vorobjov. Complexity of null-and positivstellensatz proofs. *Annals of Pure and Applied Logic*, 113(1):153–160, 2001

[13] Polynomial calculus is a proof system over polynomial equations. A derivation in polynomial calculus from initial polynomials $f_1, \ldots, f_\ell$ is a sequence of polynomials where each polynomial is either an initial one (i.e., some f_i), or it is derived from previous polynomials in the sequence using one of the inference rules

$$\frac{p \quad q}{\alpha p + \beta q}, \quad \frac{p}{xp}$$

for any polynomials p, q, coefficients α, β and variable x. The degree of the derivation is the largest degree among the polynomials in the sequence.

Exercise 4. Show that a Positivstellensatz Calculus can always be converted into a (maybe of much larger degree) Positivstellensatz proof. Deduce that whatever can be proved in one system can be proved in the other.

Sums-of-squares proofs and Lasserre hierarchy

The sums-of-squares (SOS) proof system is a weaker version of the Positivstellensatz proof, and in corresponds to the Lasserre hierarchy of positive semidefine relaxations $\{0,1\}^n$ programs. We consider as usual the system of constraints (4). A SOS proof of $p \geq 0$ is an equation

$$p = u_0 + \sum_j u_j h_j + \sum_i g_i f_i, \tag{10}$$

where u_j are sums-of-squares and g_i are arbitrary polynomials. Let us compare SOS with Positivstellensatz

- An SOS proof is essentially a Positivstellensatz proof as (5) where one is allowed to multiply at most one of the initial inequalities together;
- If there are no inequalities among the axioms, the two proof systems are equivalent.

Exercise 5. Show that the claim of Exercise 2 holds in SOS proofs too.

We already discussed that in $\mathbb{R}^n$ the Positivstellensatz is complete for refutation, but it is not implicationally complete. Instead in the boolean setting Positivstellensatz is implicationally complete, and indeed the SOS fragment is sufficient to prove that.

Theorem 6 (Implicational completeness of SOS)**.** *Consider a set of polynomial constraints*

$$\begin{aligned} &f_1 = 0, \dots, f_k = 0 \\ &x_1^2 - x_i = 0, \dots, x_n^2 - x_n = 0 \\ &h_1 \geq 0, \dots, h_m \geq 0 \end{aligned} \tag{11}$$

and p that is non negative whenever the axioms in (11) hold, there there exists a sums-of-squares proof of $p \geq 0$ from (11).

Proof hint. First observe that for any function $f : \{0,1\}^n \to \mathbb{R}$ there is a unique representation as a polynomial so that every monomial in it is multilinear (i.e. no variable is raised to a power larger than one). As a second step notice that if p and q are two polynomials over $x_1, \dots, x_n$ that compute the same function over $\{0,1\}^n$, then $p - q = \sum_i g_i(x_i^2 - x_i)$. Finally use polynomials

$$\delta_\alpha(x) = \begin{cases} 1 & \text{if } x = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

for $\alpha \in \{0,1\}^n$ to build the SOS proof by cases, showing how for each α, $p(\alpha)$ can be written as the positive multiple of one of the premises, evaluated on α. □

Geometric interpretation

We saw the SOS proof system. While this proof system could seem less expressive than Positivstellensatz, it has a natural geometric interpretation as the dual of the Lasserre hierarchy of semidefinite relaxations of integer programs. For our purposes it is more interesting to focus on the proof system, but it is important to keep in mind the geometric interpretation as well. We start with a set of polynomial inequalities $h_1 \geq 0, \dots, h_m \geq 0$ as in the Sherali-Adams relaxation. In this description we do not allow equality constraints, but we can deal with them by splitting each equality $f_i = 0$ into $f_i \geq 0$ and $f_i \leq 0$.

Let us fix $t \geq 0$ as the degree of the relaxation. As it was the case with Sherali-Adams we lift the problem to a larger space and we implicitly enforce $x_i^2 = x_i$ constraints by considering multilinearized polynomials, thus we have the moment matrix $M^t = (y_{A \cup B})$ indexed by sets $|A|, |B| \leq t/2$.

Localizing matrices. To model each constraints $h_j \geq 0$ in the relaxation we define a *localizing matrix*[14], denoted as $M^t(h_j \circ Y)$. This matrix entries are indexed by sets $|A|, |B| \leq t$. If $h_j = \sum_S \alpha_S \prod_{i \in S} x_i$, then we add linear constraints over matrix $M^t(h_j \circ Y)$ to enforce that its indexed by sets $|A|, |B| \leq t$ is equal to $\sum_S \alpha_S (y_{A \cup B \cup S})$. This linear constraints ensure that for every polynomial p, represented as a vector of coefficients, the matrix product

$$p^T M^t(h_i \circ Y) p$$

has exactly the same value as the (linearized) polynomial $h_i p^2$ evaluated on Y. Now the key part of the Lasserre relaxation is that we also want to enforce that the linearized polynomial $h_j p^2$ must be non negative for every p of degree of at most t. To do that it is sufficient to enforce that $M^t(h_j \circ Y) \succeq 0$.

[14] J. Lasserre. An explicit exact sdp relaxation for nonlinear 0-1 programs. *Integer Programming and Combinatorial Optimization*, pages 293–303, 2001

Example 7. Assume $t = 1$, $n = 2$ and let $h = x_1^2 - 3x_2$, then

$$M^1(h \circ Y) = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x_1 & x_2 \end{array} \\ \begin{pmatrix} y_{\{1\}} - 3y_{\{2\}} & y_{\{1\}} - 3y_{\{1,2\}} & y_{\{1,2\}} - 3y_{\{2\}} \\ y_{\{1\}} - 3y_{\{1,2\}} & y_{\{1\}} - 3y_{\{1,2\}} & -2y_{\{1,2\}} \\ y_{\{1,2\}} - 3y_{\{2\}} & -2y_{\{1,2\}} & y_{\{1,2\}} - 3y_{\{2\}} \end{pmatrix} \end{array}.$$

Definition 8 (Lasserre semidefinite relaxation of rank t)**.** *Let K be the set so solutions of the polynomial inequalities $h_1 \geq 0, \ldots, h_m \geq 0$. The rank t semidefinite Lasserre relaxation is the convex set $Q_t(K)$ defined as the projection of the (convex) set of solutions of the semidefinite program*

$$\begin{aligned} M^t(Y) &\succeq 0 \\ M^t(h_i \circ Y) &\succeq 0 \text{for } i = 1, \ldots, m, \end{aligned}$$

over $1 = y_\emptyset$ and $x_i = y_{\{i\}}$ for all i.

Lasserre relaxations are stronger than Sherali-Adams. The following is an example of how the constraints of the Sherali-Adams relaxation are implied by the Lasserre relaxation. Say we have the very simple initial constraint $h(x) = x_1 \geq 0$, in Sherali-Adamns we have the linearization of the constraint $x_1(1 - x_2)(1 - x_3) \geq 0$. Consider the submatrix of $M^2(h \circ Y)$ that only consider variables x_2 and x_3 is

$$\begin{array}{c} \\ 1 \\ x_2 \\ x_3 \\ x_2x_3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x_2 & x_3 & x_2x_3 \end{array} \\ \begin{pmatrix} y_{\{1\}} & y_{\{1,2\}} & y_{\{1,3\}} & y_{\{1,2,3\}} \\ y_{\{1,2\}} & y_{\{1,2\}} & y_{\{1,2,3\}} & y_{\{1,2,3\}} \\ y_{\{1,3\}} & y_{\{1,2,3\}} & y_{\{1,3\}} & y_{\{1,2,3\}} \\ y_{\{1,2,3\}} & y_{\{1,2,3\}} & y_{\{1,2,3\}} & y_{\{1,2,3\}} \end{pmatrix} \end{array}. \tag{12}$$

The Lasserre condition $M^2(h \circ Y) \succeq 0$ implies that $p^T M^2(h \circ Y) p \geq 0$ for any vector p, in particular it holds for the vector $p = (1 \quad -1 \quad -1 \quad 1)^T$. Performing the calculation yields

$$y_x - y_{xy} - y_{xz} + y_{xyz} \geq 0 \tag{13}$$

which is the constraint we had in the Sherali-Adams relaxation.

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